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SELF-SIMILAR CONDENSING FLOWS IN POROUS MEDIA*

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Abstract – Similarity solutions are obtained for the propagation of a condensation wave into an initially dry porous matrix which receives an inflow of saturated vapor due to a step increase in temperature and pressure at the boundary. The generalized Darcy (low Reynolds number) formulation of two-phase flow leads to hyperbolic/parabolic equations in which capillarity and heat conduction are suppressed in order to emphasize the shock-like behavior. Application of the $x/\sqrt{t}$ similarity transformation gives ordinary differential equations which are solved by shooting methods, using jump-balance (Rankine–Hugoniot) conditions to preserve discontinuities in saturation (quality), pressure gradient and sometimes temperature. The distribution of condensate (saturation) is wave-shaped, with a forward-facing shock on the leading side. For a small temperature difference, there is little condensate and it is nearly immobile; the saturation shock lies close to the boundary, and the outer region is described by a reduced system of equations. With increasing temperature difference, the shock moves forward into the flow and gains in strength until the medium is liquid-full behind the shock. Beyond this, the shock splits into a pair of back-to-back shocks separated by a subcooled liquid slug. The considered prototypic problem is representative of a broad class of two-phase flows which occur in energy-related and geologic applications.

NOMENCLATURE

Independent variables

x, $X = x/L$;	position;
τ, $\Tau = \tau/\tau_0$,	time $[\tau_0 = \Gamma\epsilon L/u_0]$;
θ, $\Theta = \theta/R$,	similarity variable $\theta = X/\sqrt{\Tau}$.

Dependent variables

c, $C = c\Delta T/(h_{lv})_0$,	specific heat;
h, $H = h/(h_{lv})_0$,	enthalpy $[(h_{lv})_0 = h_{lv}(P_0)]$;
k, $K = k/k_0$,	thermal conductivity;
p, $P = (p - p_0)/\Delta p$,	pressure $(\Delta p = p_1 - p_0)$;
s, $S = s/\Delta s$,	saturation: liquid fraction by volume
t, $T = (t - t_0)/\Delta t$,	temperature $(\Delta t = t_1 - t_0)$;
u, $U = u/u_0$,	Darcy velocity $(u_0 = \kappa\Delta p/L\mu_v)$;
v, $V = v/u_0$,	interstitial velocity;
κ_l, $K_l = \kappa_l/\Delta s^3$,	relative permeability of liquid;
κ_v, $K_v = \kappa_v$,	relative permeability of vapor;
ρ, $P = \rho/N\rho_0$,	density $[\rho_0 = \rho_v(p_0, t_0)]$;
ϕ, $\Phi = \phi t_0/(h_{lv})_0$,	entropy;
$\langle\rho c\rangle$, $\langle\rho C\rangle$, $= \langle\rho c\rangle/\langle\rho c\rangle_0$,	bulk specific heat, $\langle\rho c\rangle = (1-\epsilon)\rho_m c_m + \epsilon s\rho_l c_l$, $\langle\rho c\rangle_0 = (1-\epsilon)\rho_m c_m + \epsilon\Delta s\rho_l c_l$.

Physical constants

$\tilde{R}$,	gas constant;
δ,	characteristic pore dimension;
ϵ,	porosity;
κ,	permeability;
μ,	viscosity.

Dimensionless parameters

$N = p_1/p_0$,	pressure ratio;
$Pe = \dfrac{N\rho_0 u_0 h_{lv}}{\langle k\rangle\Delta t/L}$,	Peclet number;
$R = \dfrac{\mu_v}{\mu_l}\dfrac{\rho_l}{\rho_0}\dfrac{\Delta s^3}{N}$,	relative liquid mobility;
$Re = \dfrac{u_0\delta\rho_0}{\mu_v}$,	Reynolds number;
$\Gamma = \dfrac{\Delta s\rho_{lv}}{N\rho_0}$,	relative density change;
$\beta = \dfrac{\epsilon\rho_l c_l}{(1-\epsilon)\rho_m c_m}$,	specific heat ratio;
$\Delta s = \dfrac{\langle\rho c\rangle_0\Delta t}{\epsilon\rho_l(h_{lv})_0}$,	nominal liquid saturation;
$\lambda = \Delta t/t_0$,	relative temperature change.

1. INTRODUCTION

CONDENSING flows in porous media occur in a number of energy-related applications. Steam injection into oil fields produces a condensation wave which heats the oil sands and reduces crude oil viscosity. *In situ*

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combustion processes such as oil shale retorting and coal gasification are accompanied by propagating zones of evaporation and condensation. Hypothetical reactor accidents may involve boiling and condensation in fragmented debris and in porous concrete which is subjected to intense heating. Other examples arise in geothermal systems and in the containment of underground nuclear tests.

A distinctive feature of pressure-driven condensing flows in porous media is the occurrence of a sharp, wave-like saturation front. The steep gradients of a condensation front are apparent in numerical simulations such as those of Morrison [1] and Weinstein, Wheeler and Woods [2]. However, the customary integration techniques permit smearing of the saturation shocks [3] in a manner analogous to the artificial viscosity effects of numerical gas dynamics. Also, the previous studies include application specific aspects such as multiple chemical species in either the liquid or the vapor phase, which de-emphasize the generic features of condensing flows in porous media. In contrast, the present study is concerned with a more fundamental condensation wave, and our primary purpose is to describe the mathematical and physical features of the shock phenomena.

The prototype condensation problem to be considered is the one-dimensional, transient flow of a compressible pure substance. Hot, dry vapor flows into a cold, initially dry, solid matrix, therein forming condensate which flows concurrently with the vapor. Energy transfer occurs by convection and condensation, and for each fluid phase the balance between viscous and pressure forces is accounted for by the generalized Darcy law which incorporates relative permeability functions. To emphasize the shock-like behavior, capillary pressure and heat conduction are suppressed.

The parabolic/hyperbolic transport equations reduce to ordinary differential equations under the similarity transformation, $\theta = x/\sqrt{\tau}$. Since the system is only third order but has four independent boundary conditions, a saturation shock must occur. The ordinary differential equations are solved by a shooting method which uses jump-balance relations in crossing the shocks. A family of steam flows in geologic media serves to illustrate solution behavior over a broad range of the parameters. A summary of the main results is given at the end of the paper.

2. FORMULATION

The transport equations for transient, one-dimensional, two-phase, compressible flow of a pure substance in a porous medium are as follows [4]:

$$\frac{\partial}{\partial \tau}\{\epsilon s \rho_l + \epsilon(1-s)\rho_v\} + \frac{\partial}{\partial x}\{\rho_l u_l + \rho_v u_v\} = 0$$

$$\frac{\partial}{\partial \tau}\{\epsilon s \rho_l h_l + \epsilon(1-s)\rho_v h_v + (1-\epsilon)\rho_m h_m\} + \frac{\partial}{\partial x}\{\rho_l h_l u_l + \rho_v h_v u_v\} - \frac{\partial}{\partial x}\left\{\langle k\rangle \frac{\partial t}{\partial x}\right\} - \frac{Dp}{D\tau} = 0. \quad (1)$$

The subscripts l, v and m, respectively, refer to the liquid, the vapor, and the solid matrix; s is the local volume fraction of the pore space which is occupied by a liquid, ϵ is porosity, $\langle k\rangle$ is bulk thermal conductivity of the fluid saturated medium, and the other symbols have their usual meaning. The apparent velocities, u_l and u_v, represent average volumetric flow rates per unit sectional area of the medium. Darcy's law serves as a constitutive equation which relates velocity to pressure gradient in low Reynolds number flow ($Re \equiv u\delta/v$) where viscous forces are in balance with pressure forces,

$$u_v = -\kappa_v \frac{\kappa}{\mu_v}\frac{\partial p}{\partial x}$$
$$u_l = -\kappa_l \frac{\kappa}{\mu_l}\frac{\partial p}{\partial x}. \quad (2)$$

The relative permeability functions κ_v and κ_l are introduced as a means of extending Darcy's law to a two-phase flow. The utility of this approach derives from the experimental observation that κ_l and κ_v are, for a given medium, primarily functions of saturation alone [5]. The present study will make use of analytical expressions similar to those given by Scheidegger [6],

$$\kappa_l = s^3 \quad \kappa_v = 1 - s, \quad (3)$$

with the understanding that these functions are only representative of the expected behavior. In addition, it is assumed that the fluid and the solid are in local thermal equilibrium; that buoyancy forces are negligible, and that interfacial tension is accounted for implicitly through the relative permeability functions.

Thermodynamic relationships are described by the conventional analytical approximations. The liquid is incompressible; the gas is ideal, $\rho_v = p/\tilde{R}t$, and in two-phase regions, the pressure and temperature are related by the Clausius–Clapeyron equation

$$\frac{dp}{dt} = \frac{h_{lv}\rho_v}{t}\frac{\rho_l}{\rho_{lv}} \simeq \frac{p h_{lv}}{\tilde{R}t^2}\frac{\rho_l}{\rho_{lv}} \quad (4)$$

in which $h_{lv} = h_v - h_l > 0$ and $\rho_{lv} = \rho_l - \rho_v > 0$. The enthalpies h_l, h_v and h_m of the liquid, the saturated vapor and the matrix each depend linearly on temperature with respective slopes (specific heats) c_l, c_v and c_m. To eliminate secondary parameters, let $c_v = 0$, and suppose that the viscosities, μ_l and μ_v, and the matrix properties, κ and ϵ, are constants.

Regarding initial and boundary conditions, consider the case of a semi-infinite porous medium which initially contains dry saturated vapor at a temperature t_0 and corresponding saturation pressure p_0. The transient is begun by suddenly subjecting the boundary to an external environment which contains dry saturated vapor at t_1 and p_1, such that $\Delta t = t_1 - t_0 > 0$, and $\Delta p = p_1 - p_0 > 0$. Condensation occurs as the hot, high pressure fluid penetrates into the cold, porous matrix.

Upon normalization (as defined in the Nomenclature) and introduction of Darcy's law, the conservation equations are rewritten in a form which isolates the effects of phase change and emphasizes the primary dependent variables (P, T, S)

$$\frac{\partial S}{\partial \tau} - \frac{\partial}{\partial X}\left\{(P_v K_v + RK_l)\frac{\partial P}{\partial X}\right\} = -\frac{1}{\Gamma}(1 - S\Delta s)\frac{\partial P_v}{\partial \tau}$$

$$\langle PC\rangle\frac{\partial T}{\partial \tau} - H_{lv}\left\{\frac{\partial S}{\partial \tau} - R\frac{\partial}{\partial X}\left(K_l\frac{\partial P}{\partial X}\right)\right\} - \left(\frac{\Delta s\beta}{1 + \Delta s\beta}\right)RK_l\frac{\partial P}{\partial X}\frac{\partial T}{\partial X}$$

$$= Pe^{-1}\frac{\partial}{\partial X}\left(\langle K\rangle\frac{\partial T}{\partial X}\right) + \Gamma^{-1}\frac{\tilde{R}t_0}{\epsilon h_{lv}}\frac{(N-1)}{N}\frac{DP}{D\tau} \quad (5)$$

$$\langle PC\rangle = \frac{(1 + \Delta s\beta S)}{(1 + \Delta s\beta)}; \quad K_v = 1 - \Delta s S; \quad K_l = S^3$$

$$P_v = \frac{1 + (N-1)P}{N}\left\{\frac{1}{1 + \lambda T}\right\}$$

$$H_{lv} = 1 - \left(\frac{\Delta s\beta}{1 + \Delta s\beta}\right)T.$$

Here and hereafter we make the approximation that $\rho_l/\rho_{lv} \simeq 1$. The principal parameters are N, λ and β as defined in the Nomenclature; as well as the characteristic saturation $\Delta s = \langle \rho c\rangle_0 \Delta t/\epsilon\rho_l(h_{lv})_0$, which represents the amount of condensation nominally required to produce the temperature change Δt; and the relative liquid mobility $R = \mu_v\rho_l\Delta s^3/\mu_l N\rho_0$, which characterizes the relative significance of mass flow in the liquid and vapor phases. An important scaling consideration is the choice of a characteristic time τ_0

$$\tau_0 = \Gamma\frac{\epsilon L}{u_0}, \quad \Gamma = \frac{\Delta s\rho_{lv}}{N\rho_0} \quad (6)$$

which recognizes that the two-phase wave speed is very slow ($\Gamma \gg 1$) compared to a single-phase pressure wave (for which $\tau_0 = \epsilon L/u_0$, [17]), because the representative density change is $\Delta s\rho_{lv}$ rather than $N\rho_0$. Since Γ and the Peclét number, Pe, are usually very large, we can safely neglect: the time derivative of vapor density, the material derivative of pressure and the heat conduction which all appear on the RHS of the above transport equations.

The partial differential equations cannot be classified as any one of the three basic types. Superheated vapor regions are generally parabolic. Subcooled liquid regions generally appear to be elliptic because the compressibility is negligible. Two-phase regions are of a mixed parabolic/hyperbolic type [8], but become strictly parabolic as R (the liquid mobility) $\to 0$. The hyperbolic character is therefore attributed to liquid mobility and the dependence of relative permeability on saturation.

Under the similarity transformation [1, 7, 9]

$$\theta \equiv \frac{X}{\sqrt{\tau}} = \frac{x}{\sqrt{\tau}}\left(\Gamma\frac{\epsilon\mu_v}{\kappa\Delta p}\right)^{1/2} \quad (7)$$

the partial differential equations become ordinary differential equations

$$\frac{\theta}{2}S' + \{(P_v K_v + RK_l)P'\}' = 0 \quad (8)$$

$$\frac{\theta}{2}\langle PC\rangle T' + H_{lv}(P_v K_v P')' + \left(\frac{\Delta s\beta}{1 + \Delta s\beta}\right)RK_l P' T' = 0 \quad (9)$$

subject to the boundary conditions $P(0) = T(0) = 1$, $S(0) = 0$; $P(\infty) = T(\infty) = S(\infty) = 0$.

If shock-like discontinuities occur at a singular point $\theta = \theta_s$, local conservation must be enforced through the jump-balance [10] or Rankine–Hugoniot conditions. The jump-balance of momentum, roughly $[p] = \mathrm{ord}\{\rho u^2\}$, usually admits a first-order pressure jump at a shock, but the estimates $u \sim \mathrm{ord}\{\Delta p\kappa/\mu L\}$ and $\kappa \sim \mathrm{ord}\{\delta^2\}$ suggest that $[P] \sim \mathrm{ord}\{Re\ \delta/L\}$, so that $[P]$ becomes negligible once the flow has penetrated deep enough ($L/\delta \gg Re$) to be considered asymptotically self-similar (i.e., independent of Re, which is the Darcy assumption). The jump-balances of mass and energy are then

$$[\epsilon(1 - s)P_v(V_v - V_s) + \epsilon sP_l(V_l - V_s)] = 0 \quad (10)$$

$$[\epsilon(1 - s)P_v H_v(V_v - V_s) + \epsilon sP_l H_l(V_l - V_s) + (1 - \epsilon)P_m H_m(-V_s)] \quad (11)$$

$$= Pe^{-1}\left[K\frac{dT}{d\theta}\right]$$

in which $V_s = (dx/dt)/u_0$ is the interface velocity of the discontinuity, and $V_v = U_v/(1 - s)\epsilon$ and $V_l = U_l/s\epsilon$ are the so-called pore velocities or interstitial velocities of the liquid and vapor. The second law of thermodynamics must also be satisfied in crossing a shock

$$[\epsilon(1 - s)\Phi_v P_v(V_v - V_s) + \epsilon s\Phi_l P_l(V_l - V_s) + (1 - \epsilon)\Phi_m P_m(-V_s)] \quad (12)$$

$$\geq Pe^{-1}\left[\frac{K}{1 + \lambda T}\frac{dT}{d\theta}\right].$$

Since P does not change in crossing, changes in the specific entropy ϕ are calculated as $d\phi = dh/t$ or $d\phi = c_i\,dt/t$ and, hence,

$$[\Phi_i] = C_i[\ln(1 + \lambda T)]/\lambda; \quad i = l, v, m.$$

As in the differential equations, heat conduction will be ignored in the jump equation under the supposition that $Pe \gg 1$.

3. WEAKLY-SHOCKED, STRICTLY TWO-PHASE FLOW: MODERATE N

Since saturation conditions, $T = T_{sat}(P)$, are prescribed at both ends of the interval, it is reasonable to suspect that the intermediate states might also be saturated. Such strictly two-phase flows are found to occur, provided that the temperature difference (or pressure ratio) is sufficiently small, that a liquid-full condition does not occur at any cross section.

Within two-phase regions, the temperature and pressure are related by the Clausius–Clapeyron equation [now in a dimensionless form which reflects the boundary conditions: $T(P=0)=0$, $T(P=1)=1$]

$$\frac{dP}{dT} = \frac{\{1+(N-1)P\}H_{lv}}{(1+\lambda T)^2}\left\{\int_0^1 \frac{dP}{\{1+(N-1)P\}}\middle/\int_0^1 \frac{H_{lv}dT}{(1+\lambda T)^2}\right\} \quad (13)$$

and the conservation equations reduce to a third-order system of the form

$$A\begin{bmatrix} P'' \\ S' \end{bmatrix} = \mathbf{b} \quad (14)$$

subject to four independent boundary conditions,

$$P(0)=1,\ S(0)=0;\quad P(\infty)=0,\ S(\infty)=0. \quad (15)$$

The system is therefore overconstrained.

A sign change of the determinant, $\det A$, suggests the presence of a shock. By application of the boundary conditions, it can be seen that $\det A > 0$, as $\theta \to \infty$. Conversely, at $\theta = 0$ (where the equations are singular), an expansion in powers of $\theta^{1/2}$ shows that

$$S = \frac{\theta^{1/2}}{\sqrt{(-6P_0'R)}} + \ldots \quad (16)$$

$$P' = P_0'\left\{1 + \Delta s\frac{\theta^{1/2}}{\sqrt{(-6P_0'R)}}\right\} + \ldots \quad (17)$$

$$\det A = 0, \quad \text{and} \quad \frac{d}{d\theta}(\det A) < 0 \quad \text{at } \theta = 0.$$

Thus, $\det A$ must change sign in crossing the interval.

Allowance is made for a shock on the interior of the two-phase interval. Since both phases are present on both sides of the shock, $[P]=0$ implies $[T]=0$, and hence, $[\rho_i]=[H_i]=0$, $i=l,v,m$. Thus, with $Re \ll 1$ and $Pe \gg 1$, the jump-balances of mass and energy jointly require continuity of both the vapor flux and the liquid flux

$$[(1-S\Delta s)(V_v - V_s)] = 0 \Rightarrow \left[K_v\left(P' + \frac{\theta}{2}\Gamma^{-1}\right)\right] = 0 \Rightarrow [K_vP'] \simeq 0 \quad (18)$$

$$[S\Delta s(V_l - V_s)] = 0 \Rightarrow \left[RK_lP' + \frac{\theta}{2}S\right] = 0.$$

This means that there is no local phase change at the shock and that the second law is automatically satisfied through equality.

Letting 'hatted' quantities represent function values downstream of the shock, the two conditions can be combined as follows to eliminate $\hat{P}'$

$$\hat{K}_v\left\{\frac{\theta}{2} + RP'(\hat{S}^2 + S^2 + S\hat{S})\right\} + R\hat{K}_lP'\Delta s = 0. \quad (19)$$

Which, on simplification, provides a quadratic in S having only one positive, real root. Thus, for given θ, S, P, and P' there is a unique shock strength. Once $\hat{S}$ is available, $\hat{P}'$ is readily determined from the first shock condition. It is interesting to note that if $S = \hat{S}$, the above requirement reduces to the condition for $\det A = 0$. It can also be shown that $\det A$ has the necessary sign change in crossing the shock, provided that $[S] = \hat{S} - S < 0$. So, the two-phase saturation shock must face forward, although not as a consequence of entropy change.

The system of equations is solved numerically by a shooting method. For chosen values of $P'(0) = \alpha$, a three-term expansion from the singular origin is followed by rightward numerical integration, stopping at a presumed shock location θ_s. Integration is then restarted with values of $\hat{S}$ and $\hat{P}'$ determined from the shock conditions. The two shooting parameters α and θ_s are adjusted until the asymptotic boundary conditions, $P(\infty) = S(\infty) = 0$, are both satisfied.

As an illustrative family of solutions, consider the case of dry saturated steam flowing into sandstone (β

Table 1. A family of steam flows

Configuration parameters N	Δs	R	λ	Shock locations (θ_l)	θ_s	Pressure gradient P_0'	$(R \to 0)$
$N \to 1$	0	0	0				$(-1/\sqrt{\pi})$
2	0.034	0.009	0.040		0.035	−0.533	(−0.554)
5	0.084	0.083	0.098		0.435	−0.512	(−0.557)
10	0.127	0.185	0.148		0.961	−0.505	(−0.574)
15	0.161	0.292	0.180		1.17	−0.505	(−0.589)
19.5	0.184	0.371	0.202	(1.18)	1.18	−0.507	(−0.601)
100	0.359	0.924	0.360	(1.00)	1.13	−0.515	(−0.726)
1000	0.855	2.56	0.688	(0.91)	1.21	−0.525	(−2.161)
2500	1.26	4.38	0.874	(0.89)	1.28	−0.521	(∞)
5000	1.75	7.31	1.04	(0.89)	1.34	−0.510	

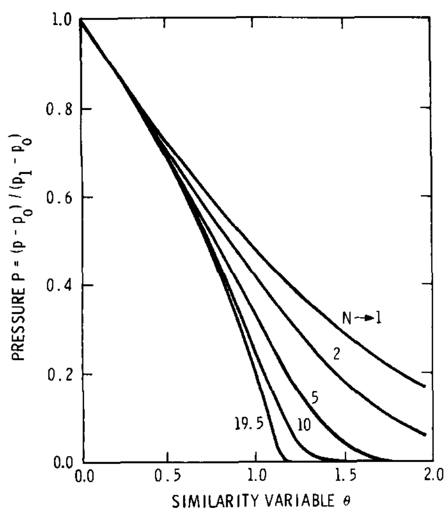

FIG. 1. Pressure profiles for various pressure ratios $N = p_1/p_0$.

$\simeq 0.6$) and suppose that the initial temperature in the medium is 530°R, corresponding to an initial pressure of $P_0 = 0.025$ atm. For such a choice of fluid, medium and initial state, Δs, R, and λ become increasing functions of N. The parameter values corresponding to particular choices of N are listed in Table 1 along with the shock location θ_s and the surface pressure gradient $P'(0)$ which were determined. Figures 1 and 2 illustrate the results.

The pressure profiles of Fig. 1 lie within a rather narrow range of θ which (based on the scaling considerations) suggests that the process is largely controlled by mass transfer in the vapor phase. As $N \to 1$ the pressure is given by $P(\theta) = \mathrm{erfc}(\theta/2)$, and for large N the pressure profiles become proximate. The local condensation rate, $(\rho_v K_v P')'$, is non-negative everywhere in the flow and reaches a maximum in the vicinity of the pressure inflection. With increasing N the saturation shock becomes more pronounced and moves forward into the flow. There is a limiting pressure ratio $\tilde{N}$ (here $\tilde{N} \simeq 19.5$) for which the pore space becomes liquid-full behind the shock.

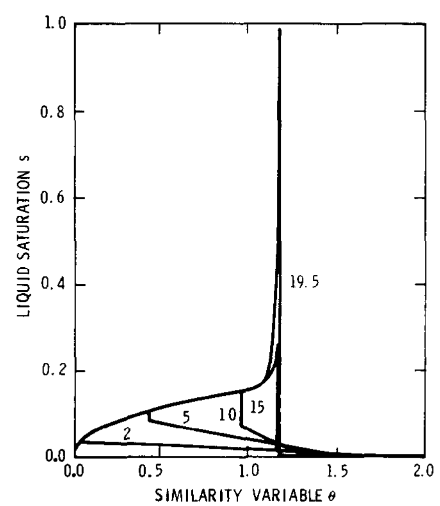

FIG. 2. Saturation profiles for various pressure ratios N.

It is generally true that the low-velocity liquid is overtaken by the shock and that the high-velocity vapor passes forward through the shock. As $N \to \tilde{N}$ the shock conditions (for $\Gamma \to \infty$) require that $P'_+ \to 0$ (suppressing a slight precursor ahead of the shock) and that $V_{l^-} \to V_s$, indicating that the wall of liquid advances with the shock velocity.

4. UNSHOCKED IMMOBILE-LIQUID LIMIT: SMALL N

When N is small, there is little condensate, it is relatively immobile, and the shock lies close to the surface. A secondary, inner scale in θ results from the disparity in phase velocities, as evident both in the saturation profiles of Fig. 2 and in the $(\theta/R)^{1/2}$ terms which appear in the inner series expansion. Such behavior suggests the existence of an outer, downstream solution complemented by a singular-perturbation boundary-layer.

In the limit of vanishing liquid mobility ($R \to 0$) the transport equations reduce to the following form

$$\frac{\theta}{2}\langle \rho C\rangle P' + \frac{\mathrm{d}P}{\mathrm{d}T} H_{lv}(\rho_v K_v P')' = 0 \qquad (20)$$

$$\frac{\mathrm{d}T}{H_{lv}} = \frac{\mathrm{d}S}{\langle \rho C\rangle} \quad \text{or} \quad S = \frac{T}{1 + \Delta s\beta(1-T)}. \qquad (21)$$

The first equation describes the transient, Darcy flow of a vapor with a large apparent compressibility because of the phase change. The second ensures energy conservation by attributing local temperature change to local condensation, whereupon T becomes an explicit function of S. With $S[T(P)]$ now available in analytical form, it is only necessary to solve a second-order, parabolic equation for the pressure. The dependence of S on P is such that the two outer boundary conditions are simultaneously satisfied if $P(\infty) = 0$, but the inner boundary conditions become incompatible. In choosing to satisfy the pressure condition $P(0) = 1$, it must also be accepted that $S(0) = 1$ and the surface can no longer be dry. This change in boundary conditions is physically reasonable because the condensate which forms near the origin must now remain in place.

The immobile liquid approximation is illustrated in Fig. 3 by a family of saturation profiles corresponding to parameter values (N, Δs, and λ) of Table 1, except that $R = 0$ is now imposed. A comparison with the liquid-mobile solutions from Fig. 2 emphasizes the typical singular perturbation behavior [11]. For small R, there is a narrow inner region where saturation gradients are steep, while the outer region remains nearly unaffected by the boundary layer and, to a good approximation, still satisfies the condition $S \to 1$ as $\theta \to 0$. As apparent in comparing the last two columns of Table 1, the immobile liquid approximation ($R \to 0$) gives a good approximation to the inflow pressure gradient.

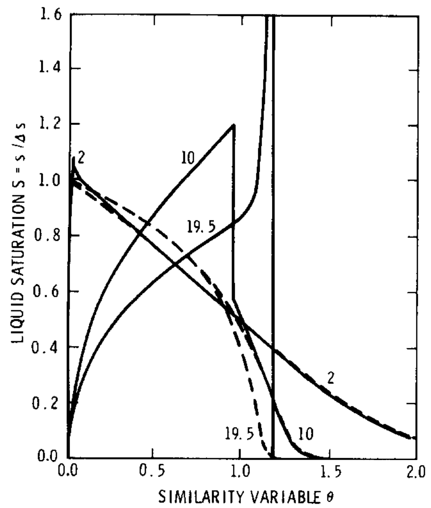

FIG. 3. Immobile liquid approximation (dotted lines) compared with shocked solutions (solid lines) for various pressure ratios $N = p_1/p_0$.

The most natural description of the boundary layer, if it exists, should involve a rescaling of the variables and matching with the outer region, a matter not pursued here. It is, however, interesting to replot the saturation profiles using $\Theta = \theta/R$ as the independent variable. For small Θ, all of the curves are nearly coincident, and as R becomes small ($N \to 1$), the shock position approaches a nonzero limit.

5. STRONGLY-SHOCKED, IMBEDDED-SLUG FLOW: LARGE N

Recall from Section 3 that for large enough N (e.g. $N \to \tilde{N} = 19.5$ in Fig. 2) the medium becomes liquid-full behind the forward-facing shock. When N exceeds $\tilde{N}$, the peak of the saturation wave broadens into a subcooled liquid zone. Within the subcooled zone $\theta_l < \theta < \theta_s$ the saturation is uniform at $S = 1/\Delta s$, but P and T are independently variable, so that the transport equations become

$$P'' = 0$$

$$\frac{\Delta s\beta}{1+\Delta s\beta}\left(RK_lP' + \frac{\theta}{2}S\frac{1+\beta}{\beta}\right)T' + \frac{1}{Pe}T'' = 0. \tag{22}$$

The pressure therefore decreases linearly in crossing the slug (i.e. velocity is uniform), and with conduction neglected (i.e. $Pe \to \infty$) the temperature must be uniform within the slug.

Allowance is now made for shocks both on the leading side and on the trailing side of the liquid slug. Letting hatted quantities refer to the slug side (wet side) of either shock and letting $[P] \equiv P(\theta_+) - P(\theta_-) = 0$, $Pe \to \infty$, and $\Gamma \to \infty$; conservation of mass and energy, respectively, require that

$$\left[RK_lP' + \frac{\theta}{2}S\right] + [\rho_v K_v P'] = 0 \tag{23}$$

$$\frac{\Delta s\beta}{1+\Delta s\beta}[T]\left(R\hat{K}_l\hat{P}' + \frac{\theta}{2}\hat{S}\frac{1+\beta}{\beta}\right) + H_{lv}[\rho_v K_v P'] = 0. \tag{24}$$

Phase change is, therefore, allowable, provided that the change in liquid flux relative to the shock is offset by the change in vapor flux and that the enthalpy of phase change accounts for the temperature jump of the mass flowing through the shock, including the mass of the solid phase. In addition, it is necessary to satisfy the second-law of thermodynamics (entropy-jump inequality)

$$\frac{\Delta s\beta}{1+\Delta s\beta}\frac{[\ln(1+\lambda T)]}{\lambda}\left(R\hat{K}_l\hat{P}' + \frac{\theta}{2}\hat{S}\frac{1+\beta}{\beta}\right) + \frac{H_{lv}}{1+\lambda T}[\rho_v K_v P'] \le 0, \tag{25}$$

which reflects the identity, $\Phi_{lv} = H_{lv}/(1+\lambda T)$. Finally, there is a temperature-jump inequality, $\hat{T} \le T$, which ensures that the slug side is not superheated.

In checking the necessity and the admissibility of temperature jumps, it is found that $T = T_{sat}[P(\theta_s)]$ everywhere within the slug. The argument consists of two main points:

(1) There must be a downward temperature jump at θ_l. Suppose to the contrary that $T = T_{sat}(P)$ at θ_{l+}. Then, upon moving rightward into the slug, T stays constant (from energy equation) while $T_{sat}(P)$ decreases as the pressure falls. So, $T_{sat}(P)$ falls below T, suggesting superheated conditions in a liquid region — a contradiction.

(2) There cannot be a temperature jump at θ_s. At any shock location, θ, the jump conditions on energy and entropy jointly require that

$$\left\{[\ln(1+\lambda T)] - \frac{[1+\lambda T]}{(1+\lambda T)}\right\} \times \left(R\hat{K}_l\hat{P}' + \frac{\theta}{2}\hat{S}\frac{1+\beta}{\beta}\right) \le 0. \tag{26}$$

Now, at θ_l the slug side is ahead of the shock, so that the script brackets above cannot be positive. Conversely, at θ_s the slug side is behind, so that the script brackets cannot be negative. Thus, a temperature jump is second-law admissible at θ_l and/or at θ_s, only if

$$R\hat{K}_l\hat{P}' + \frac{\theta}{2}\hat{S}\frac{1+\beta}{\beta} \ge 0$$

and/or (27)

$$R\hat{K}_l\hat{P}' + \frac{\theta}{2}\hat{S}\frac{1+\beta}{\beta} \le 0$$

at θ_l and θ_s, respectively. Since the first condition must be satisfied [because (1) above demands a T-jump at θ_l], and since $\theta_s > \theta_l$, $\beta > 0$, and $R\ K_l$ $\hat{P}'$ is the same at θ_l and θ_s (from continuity); it is impossible to satisfy the later inequality. There can be no temperature jump at θ_s.

In view of the above observations and the previously noted absence of an internal temperature gradient, it is concluded that $T = T_{sat}[P(\theta_s)]$ on $[\theta_{l+}, \theta_s]$.

Since temperature jump cannot occur at the leading edge, θ_s, phase change cannot occur and the energy balance requires that $\rho_v K_v P' = 0$ at θ_{s+}. The density cannot vanish; and if $\kappa_{v+} = 0$, then $S = 1/\Delta s$ on both sides, and det A does not change sign in crossing. Thus, it must be that $P'_+ = 0$, whereupon S'_+ vanishes along with all of the higher-order derivatives, ruling out downstream variation in the dependent variables. So, in view of the boundary conditions, it must also be true that

$$P = 0,\ S = 0,\ T = 0 \quad \text{at } \theta_{s+} \tag{28}$$

and it only remains to satisfy the jump-mass balance which now reads as

$$R\hat{K}_l\hat{P}' + \frac{\theta_s}{2}\hat{S} = 0 \quad \text{at } \theta_{s-}. \tag{29}$$

This is recognized as the liquid-wall condition, $\hat{V}_l = V_s$, which previously accompanied the vanishing (suppressed) precursor in the limiting case as $N \to \tilde{N}$. So, the slug solution appears to be a continuous extension of the earlier two-phase flows.

Now, consider the trailing shock which lies behind the slug at θ_l. A temperature drop, $[T] = -T(\theta_l)$, must occur here to ensure subcooled conditions throughout the slug. Using this, the jump-balances of mass and energy are alternatively combined to give both of the following expressions:

$$\rho_v K_v P' + T\frac{\Delta s\beta}{1+\Delta s\beta}\left\{RK_lP' + \left(S + \frac{1}{\Delta s\beta}\right)\frac{\theta}{2}\right\} = 0$$

$$\hat{P}' = \frac{\Delta s^3}{R}\left\{H_{lv}\left(RK_lP' + \frac{\theta}{2}S\right) - \theta T\frac{(1+\beta)}{2(1+\Delta s\beta)} - \frac{H_{lv}\theta}{2\Delta s}\right\}. \tag{30}$$

The first is a compatibility condition which must be satisfied on the upstream side of the shock. It is used to determine the shock location. The second gives $\hat{P}'$ in terms of upstream data. This is all that is needed to resume downstream integration since $\hat{P} = P$, $\hat{T} = 0$, and $\hat{S} = 1/\Delta s$.

The computational procedure for a doubly shocked flow is simple because there is only one shooting parameter. For chosen $\alpha = P'(0)$, the two-phase equations are integrated forward until the compatibility condition is satisfied, thereby determining θ_l. Then, $\hat{P}'(\theta_l)$ is calculated from the upstream data. Recalling that $P'' = 0$ in the slug and that $P(\theta_s) = 0$, the leading shock must lie at $\theta_s = \theta_l - P(\theta_l)/\hat{P}'(\theta_l)$. Then, since $\hat{P}'(\theta_s) = \hat{P}'(\theta_l)$, there is sufficient information to indicate whether or not the remaining shock condition ($\hat{V}_l = V_s$) is satisfied at θ_s; this being the sole criterion for iterative adjustment of α.

Doubly-shocked solutions are illustrated by the pressure and saturation profiles of Figs. 4 and 5. With increasing N, the slug broadens (see Table 1), and the pressure rises at θ_l to overcome the viscous drag on the slug. Both the liquid and the vapor overtake the condensation shock. At the leading edge θ_s the liquid velocity matches the shock speed. However, on both sides of θ_l the liquid velocity exceeds the local shock speed, providing a mechanism for the timewise growth (self-similar stretching) of the slug.

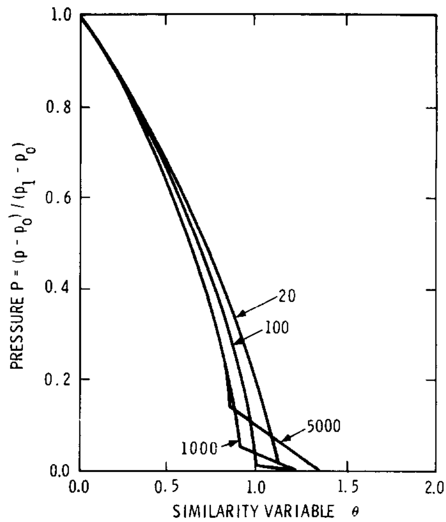

FIG. 4. Pressure profiles for various pressure ratios $N = p_1/p_0$.

6. DISPERSIVE MECHANISMS

Smearing of saturation shocks by the capillarity of a porous medium is analogous to viscous smearing of gas-dynamic shocks, as discussed previously [5, 6] for the classical Buckley–Leverett problem. If a finite capillary pressure were explicitly included in the present model, s_{xx} would appear along with s_x and s_t in the continuity equation, the system would become fourth order with a proper number of boundary

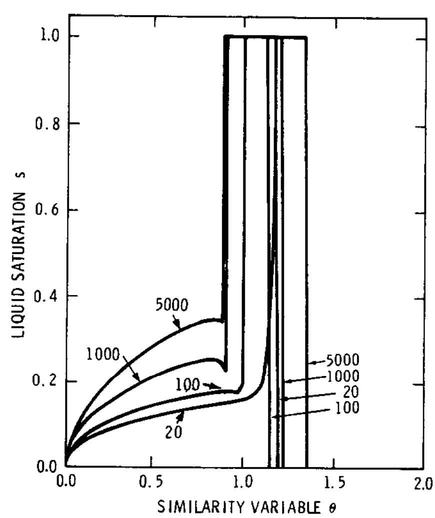

FIG. 5. Saturation profiles for various pressure ratios N.

conditions, and there would be no mathematical necessity for a shock (although computational difficulties may still persist). Although the mechanisms of capillarity and hydrodynamic dispersion preclude a true discontinuity in saturation, these smearing effects should be local, as in gas dynamic shock. Likewise, the effects of macroscopic fingering instability (which might occur in two-phase drive of a liquid slug) should be negligible because of the stabilizing influence of volumetric contraction at the moving condensation front [12].

Smearing of thermal shock by heat conduction is assessed by reformulation of the current problem under the suppositionof asymptotically large, but now finite, Peclét number. Within two-phase regions, the perturbation (due to $\epsilon = 1/Pe \ll 1$) is regular [11] because the T'' conduction term in the energy equation can be grouped (using Clausius–Clapeyron) with the stronger P'' terms and, hence, the order of the equation is not changed. However, in the subcooled liquid slug, the addition of T'' raises the order of the energy equation. The temperature distribution then remains uniform across the slug, except within a thermal boundary layer which replaces the temperature jump at θ_l. The singular perturbation [11] of the slug solution shows that the boundary layer thickness is of order $1/Pe$ and that the temperature gradient at θ_l is, to the first order

$$T' = [T]Pe\frac{\Delta s\beta}{1+\Delta s\beta}\left(R\hat{\kappa}_l\hat{P}' + \frac{\theta}{2}\hat{S}\frac{1+\beta}{\beta}\right), \quad (31)$$

in which $[T]$ is now interpreted as the temperature change in crossing the boundary layer. In writing the jump energy balance for the saturation shock which still persists at θ_l, temperature jump is now omitted, but a conduction term $[T'/Pe]$ is included, thereby arriving at identically the same shock conditions. As $Pe \rightarrow \infty$, the solution is, therefore, the same as before.

Regarding second-law considerations, it is noteworthy that a necessary condition for existence of the (exponential) thermal boundary layer at high Pe is that

$$\left(R\hat{\kappa}_l\hat{P}' + \frac{\theta}{2}\hat{S}\frac{1+\beta}{\beta}\right) > 0. \quad (32)$$

Since this criterion is identical to the entropy-jump inequality (27a) for a thermal shock, it follows that the second-law becomes extraneous when heat conduction is included. An analogous situation occurs in shocked gas flows where the inviscid equations must be supplemented by entropy considerations, but the complete Navier–Stokes equations (including viscous smearing through u'') are self-sufficient.

7. SUMMARY

The considered prototype problem retains only the essential features of a condensing flow in a porous medium: concurrent gas/liquid mass transfer, convective energy transfer, and condensation due to fluid/solid energy exchange. Since capillarity and heat conduction are suppressed, the transport equations are of a mixed parabolic/hyperbolic type which demands shock-like jumps in saturation and temperature. In the considered case of a dry saturated-vapor inflow there is a singularity at the injection surface which suppresses the inner structure in order to focus on a representative, but relatively simple, outer structure. In a subsequent paper [13], we consider the behavior under other boundary conditions, particularly those which result in imbedded regions of superheated vapor, as well as the case of a partially wet or fully wet far field.

Physical characteristics of the flow are strongly dependent on the magnitude of the temperature difference ΔT (parameterized by the pressure ratio, N), since it determines the nominal amount of condensation, Δs.

1. When ΔT is small, there is little condensation and the liquid is nearly immobile. The flow contains a weak saturation shock which lies close to the inflow boundary. The outer downstream region is second-order parabolic and resembles a single-phase vapor flow with a large (phase-change) compressibility.
2. So long as ΔT is moderate, the amount of condensate is insufficient to cause liquid-blockage of the pore space. At the saturation shock both phases are present on both sides, and the jump conditions require that: the shock faces forward, that there be no local phase change, that there be no local temperature jump. The low-mobility liquid is overtaken by the shock while the high-mobility vapor passes forward through the shock. Each mass flow is continuous in the shock frame, as in the classical Buckely–Leverett flow [5, 6].
3. At large ΔT, the liquid-full condition prevails over an interval in which the (incompressible) flow velocity and the temperature are uniform. The leading edge of this liquid slug is simply a wall of liquid, and the medium is undisturbed ahead (for $\Gamma \rightarrow \infty$). This full strength forward-facing shock is the continuous extension of the previous two-phase shock and, accordingly, there is no local phase change. On the trailing side of the slug there is a backward-facing saturation shock. Both liquid and vapor overtake the shock, and there is a temperature jump and some local condensation.

A family of steam flows in geologic media serves to illustrate solution behavior over a broad range of the parameters. It is noteworthy that the penetration depth $\theta_s \simeq 1$ and the pressure gradient $P'(0) \simeq -0.5$, are very weak functions of the parameters, because the scaling considerations absorb the first-order dependency (as also checked for other fluid/solid systems). Thus, the scaling considerations and the computational results have considerable generality in estimating penetration depth, flow rates, and flow structure for condensing flows in initially dry porous media.

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ECOULEMENTS SELF-SIMILAIRES AVEC CONDENSATION DANS LES MILIEUX POREUX

Résumé—On obtient des solutions similaires pour la propagation de l'onde de condensation dans une matrice poreuse initialement sèche et qui reçoit un flux de vapeur saturée dû à un accroissement en échelon de température et de pression à la frontière. La formulation généralisée de Darcy (faible nombre de Reynolds) d'un écoulement diphasique conduit à des équations hyperboliques/paraboliques dans lesquelles la capillarité et la conduction thermique sont supprimées de façon à dégager le comportement semblable à un choc. L'application de la transformation en $X/\sqrt{t}$ donne des équations aux dérivées partielles qui sont résolues en utilisant un bilan de saut (Rankine–Hugoniot) pour traiter des discontinuités de saturation (qualité) de gradient de pression et de température éventuellement. La distribution du condensat (saturation) présente un front. Pour une faible différence de température, il y a un faible condensat proche de l'immobilité; le choc de saturation est proche de la frontière et la région externe est décrite par un système d'équations réduit. Lorsque la différence de température augmente, le choc se déplace en avant et gagne en intensité jusqu'à ce que le milieu soit plein de liquide derrière le choc. Le choc se déplace en avant et gagne en intensité jusqu'à ce que le milieu soit plein de liquide derrière le choc. Le choc se sépare en deux chocs dos-à-dos, séparés par un noyau de liquide sous-refroidi. Le problème considéré est représentatif d'une grande classe d'écoulements diphasiques qui sont rencontrés dans les applications liées à l'énergie et à la géologie.

ÄHNLICHE KONDENSIERENDE STRÖMUNGEN IN PORÖSEN MEDIEN

Zusammenfassung—Für die Ausbreitung einer Kondensations-Welle in eine anfänglich trockene poröse Matrix, in die durch ein sprunghaftes Ansteigen von Temperatur und Druck an ihrem Rande gesättigter Dampf einströmt, werden Ähnlichkeitslösungen gewonnen. Die verallgemeinerte Darcy-Formulierung (kleine Reynolds-Zahl) für Zwei-Phasen-Strömung führt auf hyperbolisch/parabolische Gleichungen, in denen zur Betonung des stoßartigen Verhaltens Kapillarwirkung und Wärmeleitung vernachlässigt werden. Die Anwendung der $X/\sqrt{t}$—Ähnlichkeits-Transformation führt zu gewöhnlichen Differential-Gleichungen, die durch Monte-Carlo-Verfahren gelöst werden; hierbei werden Sprungbedingungen (Rankine–Hugoniot) verwendet, um Unstetigkeiten der Sättigung (Dampfgehalt) des Druckgradienten und manchmal der Temperatur zu erhalten. Die Kondensat-Verteilung (Sättigung) ist wellenförmig, mit einer Stoßfront an der Vorderseite. Bei einer kleinen Temperatur-Differenz entsteht wenig Kondensat, und es ist fast unbewegt; der Sättigungs-Stoß liegt nahe am Rand, das Gebiet außerhalb wird durch ein reduziertes Gleichungssystem beschrieben. Mit zunehmender Temperatur-Differenz bewegt sich der Stoß vorwärts in die Strömung und nimmt dabei an Stärke zu, bis hinter dem Stoß gesättigte Flüssigkeit vorliegt. Danach teilt sich der Stoß auf in ein Paar durch einen unterkühlten flüssigen Pfropfen getrennter Stöße. Das betrachtete beispielhafte Problem steht für eine große Gruppe von Zwei-Phasen-Strömungen, die im Bereich der Energietechnik und der Geologie Anwendung finden.

АВТОМОДЕЛЬНЫЕ РЕШЕНИЯ ПРИ КОНДЕНСАЦИИ В ПОРИСТЫХ СРЕДАХ

Аннотация — Получены автомодельные решения для случая распространения волны конденсации в первоначально сухой пористой матрице, в которую начинает поступать насыщенный пар, в результате скачкообразного увеличения температуры и давления на границе. Обобщенная формулировка закона Дарси (малое число Рейнольдса) для двухфазного потока позволяет получить гиперболическое и параболическое уравнения, в которых проницаемость и теплопроводность исключены для отражения скачкообразного характера процесса. Применение преобразования подобия $x/\sqrt{t}$ приводит к обыкновенным дифференциальным уравнениям, которые решены методом «пристрелки», используя условия Ренкина–Гюгонио на скачках насыщения, градиента давления и температуры. Распределение конденсата происходит волнообразно с разрывом на переднем фронте. При небольшой разности температур доля конденсата незначительна и он почти неподвижен; конденсат располагается у поверхности тела, а внешняя область течения описывается усеченной системой уравнений. При увеличении разности температур скачок уплотнения перемещается в сторону потока и усиливается до тех пор, пока пространство за скачком полностью не заполнится жидкостью. Вне этой области ударная волна распадается на пару ударных волн, разделенных слоем недогретой жидкости. Рассматриваемая задача характерна для широкого класса двухфазных течений, встречающихся в энергетике и геологии.